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A natural approach to extended Newtonian gravity: tests and predictions across astrophysical scales

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ABSTRACT

In the pursuit of a general formulation for a modified gravitational theory at the non-relativistic level and as an alternative to the dark matter hypothesis, we construct a model valid over a wide variety of astrophysical scales. Through the inclusion of Milgrom's acceleration constant into a gravitational theory, we show that very general formulas can be constructed for the acceleration felt by a particle. Dimensional analysis shows that this inclusion naturally leads to the appearance of a mass-length scale in gravity, breaking its scale invariance. A particular form of the modified gravitational force is constructed and tested for consistency with observations over a wide range of astrophysical environments, from solar system to extragalactic scales. We show that over any limited range of physical parameters, which define a specific class of astrophysical objects, the dispersion velocity of a system must be a power law of its mass and size. These powers appear linked together through a natural constraint relation of the theory. This yields a generalised gravitational equilibrium relation valid for all astrophysical systems. A general scheme for treating spherical symmetrical density distributions is presented, which in particular shows that the fundamental plane of elliptical galaxies, the Newtonian virial equilibrium, the Tully-Fisher and the Faber-Jackson relations, as well as the scalings observed in local dwarf spheroidal galaxies, are nothing but particular cases of that relation when applied to the appropriate mass-length scales. We discuss the implications of this approach for a modified theory of gravity and emphasise the advantages of working with the force, instead of altering Newton's second law of motion, in the formulation of a gravitational theory.

Key words: gravitation – galaxies: kinematics and dynamics – galaxies: general.

1 INTRODUCTION

The dynamical mass to light ratios derived for spiral galaxies are usually much greater than expected for their stellar components. This is often interpreted as indicating the gravitational dominance of hypothetical dark matter. Alternatively, one could argue that the discrepancy between dynamical mass and baryonic mass is telling us that the Newtonian law of gravity is not the one governing the dynamics. In particular, the MOdified Newtonian Dynamics (MOND) proposed by Milgrom (1983b) has been proven to be successful in explaining how galaxies rotate, without any dark matter (see e.g. Sanders & McGaugh 2002, for a review).

Recently, the range of astrophysical problems treated under the MOND hypothesis has increased significantly. Abundant recent publications on velocity dispersion measurements for stars in the local dwarf spheroidal galaxies, the extended and flat rotation curves of spiral galaxies, the large dispersion velocities of galaxies in clusters, the gravitational lensing due to massive clusters of galaxies, and even the cosmologically inferred matter content for the universe, have been successfully modelled under MOND. These, not as indirect evidence for the existence of a dominant dark matter component, but as direct evidence for the failure of the current Newtonian and general relativistic theories of gravity, in the large scale or low acceleration regimes relevant for the above (see Sanders & Noordermeer 2007; Nipoti et al. 2007; Famaey et al. 2007; Gentile et al. 2007; Tiret et al. 2007; Sánchez-Salcedo et al. 2008; Bekenstein 2004; Capozziello et al. 2007; Sobouti 2007; Mendoza & Rosas-Guevara 2007, for recent examples). MOND has proved successful on many astrophysical situations, though difficult on others (see e.g. Milgrom 2008, 2009b; Bekenstein 2006; Zhao 2005, for a good review on these points).

The key feature of Milgrom's Modified Newtonian Dynamics is the introduction of a fundamental acceleration scale $a_0 = 1.2 \times 10^{-10} \mathrm{m\,s^{-2}}$ (see e.g. Milgrom 2008) into gravitation. The introduction of a_0 can alternatively be regarded as grounded upon direct empirical evidence, as the

observed dynamics of large spiral discs attest. Additionally, non-relativistic gravity due to a point mass M, results in a force on any test particle such that it is pulled towards M with an acceleration a. Fundamentally, Newton's constant of gravity G completes the description of the problem and so, by means of Buckingham's theorem of dimensional analysis (cf. Sedov 1959), the absolute value of the attractive acceleration felt by the test particle located at a distance r from the point mass M is given by

$$a = a_0 \, f(x), \qquad (1)$$

where

$$x := l_{\mathrm{M}}/r, \qquad (2)$$

with,

$$l_{\mathrm{M}} := \left(\frac{GM}{a_0} \right)^{1/2}. \qquad (3)$$

The acceleration expressed in equation (1) converges to Newton's gravitational acceleration when the function $f(x) = x^2$ and to MOND's acceleration when $f(x) = x$. These two examples of functions f represent the gravitational approach to an extended non-relativistic theory of gravity. The main problem is how to find a function $f(x)$ which, for the appropriate limits, converges to the Newtonian and MONDian regimes.

Hernandez et al. (2010) showed that the Bekenstein (2004) function $f(x) = x + x^2$ serves quite well when applied to dwarf spheroidal (dSph) galaxies and to the rotation curves at large radii of spiral galaxies, but is inconsistent with measured limits on departures from Newtonian gravity at solar system scales. Also, Famaey & Binney (2005) showed that this particular form of the function $f(x)$ does not work well when applied to our Galaxy. Despite the fact that this prescription has the corresponding limits as expected ($f(x) \to x$ when $x \ll 1$ and $f(x) \to x^2$ when $x \gg 1$), a more general function must be constructed.

Note that pure dimensional analysis, with the introduction of an acceleration scale a_0, determines exactly the dimensional form that the acceleration must have. In very general terms, it also shows that the introduction of a_0 means that gravity has a characteristic mass-length scale l_{M} which makes possible the construction of equation (1). With all these, the acceleration turns out to be a function of the variable x only, which as we will show later, gives a robust way of working with an extended theory of gravity at the non-relativistic level.

It is important to note that Milgrom has already introduced the length l_{M} (see e.g. Milgrom 1983b,a, 1986; Milgrom & Sanders 2008; Milgrom 2008, where it appears as a transition radius). In these studies it is shown that this mass-length scale serves as a transition point where the MONDian regime passes to the Newtonian one. Milgrom & Sanders (2008) stressed the points that a mass distribution whose length is much greater than its associated mass-length l_{M} is in the MONDian regime (since $x \ll 1$) and a mass distribution whose length is much smaller than its mass-length scale is in the Newtonian regime (since $x \gg 1$). The case $x = 1$ can roughly be thought of as the point where the transition from the Newtonian to the MONDian regimes occurs.

We now show that there is a connection between this approach and the one commonly used in the implementation of the MOND theory. The physical form of MOND is given by (see e.g. Bekenstein & Milgrom 1984; Milgrom 2001) an Aquadratic Lagrangian (AQUAL) and so, its variation reproduces the equation of motion. For relevant symmetries in the problem, this approach gives the important result that the absolute value of the acceleration felt by a test particle in the presence of a point mass is given by

$$a \; \mu \left(a/a_0 \right) = \left| \nabla \phi_{\mathrm{N}} \right| = \frac{GM}{r^2}. \qquad (4)$$

In this equation, the Newtonian scalar potential is represented by ϕ_{N} and the interpolation function $\mu(a/a_0)$ is such that $\mu(a/a_0) = 1$ in the Newtonian limit, which corresponds to $a \gg a_0$ and $\mu(a/a_0) = a/a_0$ in the MONDian regime, with $a \ll a_0$. Equating the acceleration in relation (4) with that of equation (1), it follows that

$$\mu(a/a_0) = \frac{x^2}{f(x)}, \qquad (5)$$

which implicitly shows that the MOND formalism can be equivalently expressed through the modification of the gravitational force (1). We note here that the MOND formulation (4) refers to a modification of the dynamical sector of the theory, whereas equation (1) is completely based on the modification of the gravitational force. Both are operationally equivalent formulations. The MOND formulation has always been tackled through dynamical modifications. However, we show in this article that there are many advantages when choosing the modification in the gravitational sector. Therefore we only use the constant a_0 for consistency with the dynamical modifications. It is important to emphasise that in the gravitational modifications it is more natural to frame the problem in terms of the mass-length scale l_{M} defined in equation (3). Furthermore, it is the use of dimensional analysis which tells us the very important fact that the dimensionless force $f(x)$ in equation (1) only depends on the ratio l_{M}/r.

The article is organised as follows. Section 2 introduces a particular form of the function $f(x)$. This is used in the subsequent sections for applications in different astrophysical environments, from solar system to galaxy cluster scales. Finally, in Section 4 we discuss the advantages of such a general function $f(x)$.

2 THE FORCE MODEL

The dimensionless gravitational force $f(x)$ in equation (1) felt by a given test particle must be analytic, and as such it can be written as

$$f(x) = \sum_{n=-\infty}^{n=\infty} c_n x^n. \qquad (6)$$

We now show how to obtain a reasonable $f(x)$ by simple exploration of the Newtonian and MONDian regimes. First

of all, notice that in the Newtonian and deep MOND limits, the function $f(x)$ is such that $f(x) = c_{\mathrm{N}}\, x^2$ and $f(x) = c_{\mathrm{M}}\, x$ respectively, with $c_{\mathrm{N}} = c_{\mathrm{M}} = 1$. We now focus on the Newtonian $x \gg 1$ regime and explore an expansion about that limit of the form

$$\begin{aligned}\left(\frac{a}{a_0}\right) &= \left(\frac{l_{\mathrm{M}}}{r}\right)^2 + \left(\frac{l_{\mathrm{M}}}{r}\right) + \left(\frac{l_{\mathrm{M}}}{r}\right)^0 + \left(\frac{l_{\mathrm{M}}}{r}\right)^{-1} + \ldots, \\ &= \left(\frac{l_{\mathrm{M}}}{r}\right)^2 \left\{1 + \left(\frac{l_{\mathrm{M}}}{r}\right)^{-1} + \left(\frac{l_{\mathrm{M}}}{r}\right)^{-2} + \left(\frac{l_{\mathrm{M}}}{r}\right)^{-3} + \ldots\right\}, \\ &= x^2 \left(1 + x^{-1} + x^{-2} + x^{-3} + \ldots\right).\end{aligned} \tag{7}$$

The limit $x \to \infty$ gives the Newtonian acceleration, so taking into account all the terms of the geometric series for $x > 1$ then,

$$\left(\frac{a}{a_0}\right) = x^2 \left(1 + \frac{1}{x-1}\right) = \frac{x^3}{x-1}. \tag{8}$$

We now put special emphasis on the MONDian $x \ll 1$ regime and explore the corresponding expansion, given by

$$\begin{aligned}\left(\frac{a}{a_0}\right) &= \left(\frac{l_{\mathrm{M}}}{r}\right) + \left(\frac{l_{\mathrm{M}}}{r}\right)^2 + \left(\frac{l_{\mathrm{M}}}{r}\right)^3 + \left(\frac{l_{\mathrm{M}}}{r}\right)^4 + \ldots, \\ &= \left(\frac{l_{\mathrm{M}}}{r}\right) \left\{1 + \left(\frac{l_{\mathrm{M}}}{r}\right) + \left(\frac{l_{\mathrm{M}}}{r}\right)^2 + \left(\frac{l_{\mathrm{M}}}{r}\right)^3 + \ldots\right\}, \\ &= x \left(1 + x + x^2 + x^3 + \ldots\right).\end{aligned} \tag{9}$$

Note that the deep MOND regime is obtained in the limit $x \to 0$, and so the geometric series of equation (9) for $x < 1$ gives

$$\left(\frac{a}{a_0}\right) = \frac{x}{1-x}. \tag{10}$$

Equation (7) can be thought of as the series for the negative powers of relation (6) and equation (9) as the one for the positive powers of the same relation. The interesting thing to note is that both of them can be analytically continued for all values of x. For the limit cases, the minus sign on the denominator of both equations (8) and (10) can be changed for a positive sign.

Since we are interested in the complete analytic series let us propose a general acceleration formula given by the addition or substraction of equations (8) and (10) as follows:

$$\left(\frac{a}{a_0}\right)_{\pm} = \frac{x \pm x^3}{1 \pm x}. \tag{11}$$

Note that this last equation tends to the Newtonian acceleration regime when $x \to \infty$ and to the MONDian acceleration limit when $x \to 0$. In fact, due to the symmetry of the numerator and denominator of equation (11), a more general relation can be postulated:

$$\left(\frac{a}{a_0}\right)_{\pm} = x\, \frac{1 \pm x^{n+1}}{1 \pm x^n}. \tag{12}$$

This satisfies the Newtonian and MONDian acceleration limits for $x \to \infty,\ 0$ respectively. Note also that the case $n = 1$ with a minus sign is the same as two times the case $n = 0$ with a plus sign, and both correspond to the *Bekenstein ground state* acceleration formula (Bekenstein 2004). This has proved to be useful for the dynamical modelling of dSph galaxies (Hernandez et al. 2010), but not for our own Galaxy (Famaey & Binney 2005).

The acceleration function (12) has no singularities, since according to l'Hôpital's rule, $a/a_0 \to (n+1)/n$ as $x \to 1$. In fact, to see this directly, notice that for the minus sign it follows from equation (12) that

$$\begin{aligned}\left(\frac{a}{a_0}\right)_{-} &= x\, \frac{(1-x)(1+x+x^2+x^3+\ldots+x^n)}{(1-x)(1+x+x^2+\ldots+x^{n-1})}, \\ &= x\, \frac{(1+x+x^2+x^3+\ldots+x^n)}{(1+x+x^2+\ldots+x^{n-1})}.\end{aligned} \tag{13}$$

For further applications we note that the right hand side of equation (12) with a minus sign can be Taylor expanded as follows:

$$\left(\frac{a}{a_0}\right)_{-} = x + x^{n+1} - x^{n+2} + x^{2n+1} - x^{2n+2} + \ldots, \tag{14}$$

for $x < 1$, and

$$\left(\frac{a}{a_0}\right)_{-} = x^2 - x^{1-n} + x^{2-n} - x^{1-2n} + x^{2-2n} + \ldots, \tag{15}$$

for $x > 1$. Choosing the positive sign we obtain:

$$\left(\frac{a}{a_0}\right)_{+} = x - x^{n+1} + x^{n+2} + x^{2n+1} - x^{2n+2} + \ldots, \tag{16}$$

for $x < 1$, and

$$\left(\frac{a}{a_0}\right)_{+} = x^2 + x^{1-n} - x^{2-n} - x^{1-2n} + x^{2-2n} + \ldots, \tag{17}$$

for $x > 1$. This shows that the Newtonian and MONDian regimes are reached in the correct limit regardless of the value of n.

The extreme limiting case of $n \to \infty$ corresponds to the function

$$\left(\frac{a}{a_0}\right)_{\mathrm{e}} = \begin{cases} x, & \text{for } 0 \leqslant x \leqslant 1 \quad \text{(MONDian regime)}. \\ x^2, & \text{for } x \geqslant 1 \quad \text{(Newtonian regime)}. \end{cases} \tag{18}$$

This acceleration formula is of no use due to the discontinuity on the first derivative at $x = 1$, but serves as a reference to understand that the real acceleration must smoothly pass from the Newtonian to the MONDian regime. Also, as noted by Milgrom & Sanders (2008), the point where $x = 1$ represents approximately the transition from the Newtonian to the MONDian regimes. In the proposed model, this is strictly valid in the extreme case, with $n \to \infty$. This point is relevant, as any function moving away from the value $a = a_0$ at $x = 1$ seems to have a better chance at modelling a more real astrophysical situation, since it will smoothly transit from the MONDian regime to the Newtonian one.

Figure 1 shows a plot of a/a_0 as a function of x for

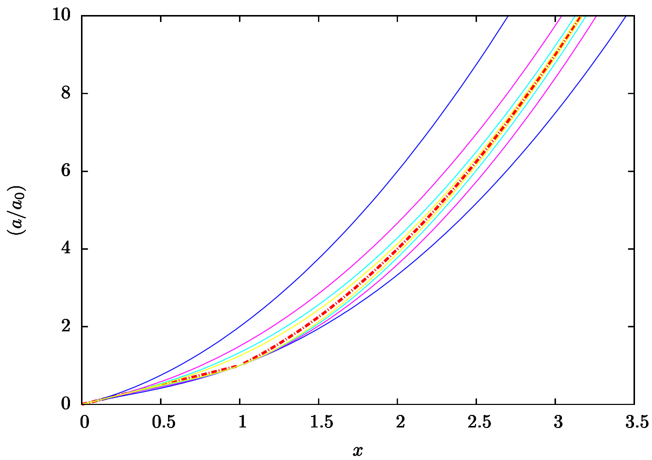

Figure 1. The figure shows the acceleration function a in units of Milgrom's constant a_0 as a function of the parameter x. The thick dash-dot curve is the extreme limiting value $n \to \infty$, i.e. $a/a_0 = x$ for $x \leqslant 1$ and $a/a_0 = x^2$ for $x \geqslant 1$. The curves above and below this extreme acceleration line represent values of $n = 4,\ 3,\ 2,\ 1$, for the minus and plus signs of equation (12) respectively. The extreme limiting curve has a kink at $x = 1$ and is of no physical interest due to the undefined derivative at that point.

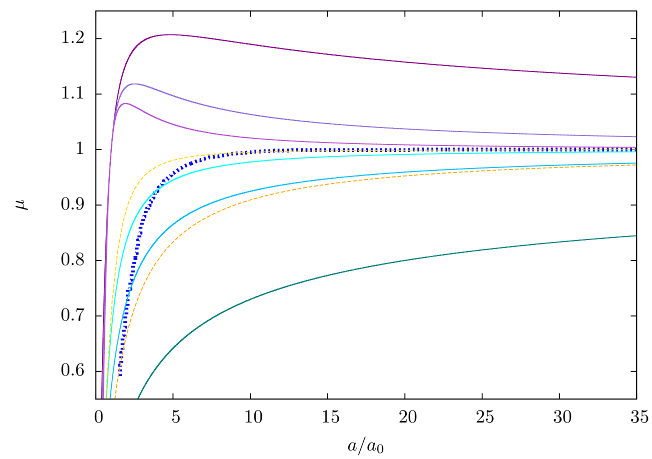

Figure 2. The figure shows MOND's interpolation function $\mu(a/a_0)$ for our Galaxy (dotted curve) as inferred by Famaey & Binney (2005). From bottom to top (not counting the dotted curve) the plots represent the following models: (1) Bekenstein ground state acceleration model (Bekenstein 2004), which corresponds to our equation (12) with $n = 1$ and a minus sign. (2) The interpolation formula $\mu(\chi) = \chi/(1+\chi)$ (Famaey & Binney 2005, dashed). (3) $n = 2$ for a minus sign. (4) $n = 3$ for a minus sign. (5) The interpolation formula $\mu(\chi) = \chi/\left(1+\chi^2\right)^{1/2}$ (Milgrom 1983b, dashed). (6) $n = 3$, plus sign. (7) $n = 2$, plus sign. (8) $n = 1$, plus sign.

various values of n and choice of signs in equation (12). From this figure it is seen that the curves with $n \sim 3-4$ are very close to the extreme limiting case, but preserve a good soft transition region between the MONDian and Newtonian regimes. However, it is through fits with observations that an optimal real number n is to be calibrated. In fact, the fit with observations must also give us a way to decide between the plus or minus sign in equation (12).

In order to fix the parameter n and the choice of sign in equation (12), we construct different corresponding MOND interpolation functions $\mu(a/a_0)$ for this acceleration formula and compare them with the best model for our Galaxy presented by Famaey & Binney (2005). To do that, we must substitute equations (12) and (1) into (5), yielding

$$\mu(a/a_0) = x\,\frac{1 \pm x^n}{1 \pm x^{n+1}}. \qquad (19)$$

The function $x(a/a_0)$ that appears in equation (19) is obtained by solving numerically equation (12) for a fixed value of n. Figure 2 shows that the best fit to the optimal $\mu(a/a_0)$ obtained by Famaey & Binney (2005) for our Galaxy, is reproduced with the minus sign and with $n = 3.13 \approx 3$. The effective gravitational acceleration formula is hence chosen as

$$\left(\frac{a}{a_0}\right) = f(x) = x\,\frac{1-x^4}{1-x^3} = x\,\frac{1+x+x^2+x^3}{1+x+x^2}. \qquad (20)$$

Given observational errors, and uncertainties in the mass to light ratios and their radial variability in our galaxy, the constrains of Figure 2 are subject to considerable uncertainties. As such, we intend to show mearly that functions of the proposed family are clearly consistent with available estimates of $\mu(a/a_0)$, in this rather uncertain parameter range. In the following section we show this particular $f(x)$ as consistent with the much more stringent constraints available and the quasi–Newtonian scale of the solar system. Also, simple explanations for the observed structural relations of elliptical and dwarf spheroidal galaxies will be shown to appear naturally in Section 3. Note that for this particular case, the function $\mu(a/a_0)$ has an analytic solution, since the function $x(a/a_0)$ from equation (12) with $n = 3$ is the root of a fourth order polynomial in x. However, due to its complicated form, we omit it here. It is interesting that for this value of n, the expansion about $x \gg 1$ begins with the Newtonian term, and then skips the following two terms according to equation (15). This guarantees that dynamics will remain extremely close to Newtonian for a large range of values of $x > 1$ and sheds light on the extended Newtonian character of the force of gravity. Similarly, for $x \ll 1$, the leading term of equation (14) gives the deep MOND regime x with the following one being x^4. The absence of the x^2 and x^3 terms implies that physics close to the deep MOND regime will not present any strong variations for a considerable range of values in $x < 1$.

We also note that the acceleration function (1) is such that Newton's theorems, i.e. the acceleration field at distance r from the centre of a spherical system depends only on the total mass $M(r)$ interior to r, while external shells result in no force, are valid for any analytic function (6) which depends on the parameter x only. In order to see this, suppose that $a \propto x^p$ with p an integer number. Assume that the test particle is placed at position r inside a spherically symmetric shell. If we trace a cone with solid angle $\delta\Omega$ and vertex at the test particle, the shell is intersected at two opposite points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. The masses δM_1 and δM_2 contained within the solid angle $\delta\Omega$ at these points keep the proportion $\delta M_2/\delta M_1 = (r_2/r_1)^2$. This relation means that $(\delta M_1/r_1^2)^{p/2} = (\delta M_2/r_2^2)^{p/2}$, and so $\delta x^p(r_1) = \delta x^p(r_2)$. In

other words, the acceleration exerted by the outer shell at position r cancels out. Since we can do this for any integer p, it follows that any analytical function of x (cf. equation (6)) guarantees Newton's theorems. This is an encouraging property of the acceleration (1).

We now extend equation (1) to a vector form. As noted above, the AQUAL formulation is satisfied by our model and as such its field equation (Poisson's generalisation formula) is given by (Bekenstein & Milgrom 1984)

$$\nabla \cdot \left[\mu(a/a_0) \, \nabla \phi \right] = 4\pi G \rho = \nabla^2 \phi_{\mathrm{N}}, \tag{21}$$

where the scalar potential ϕ satisfies the condition $\boldsymbol{a} = -\nabla\phi$. For systems with a high degree of symmetry, Bekenstein & Milgrom (1984) showed that

$$\left(\frac{\boldsymbol{a}}{a_0} \right) \mu\left(a/a_0\right) = -\frac{1}{a_0} \nabla \phi_{\mathrm{N}} = -\frac{G}{a_0} \int \frac{(\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \, \rho(\boldsymbol{r}') \, \mathrm{d}V', \tag{22}$$

where we can now express the interpolation function μ through expression (5), and $\rho(\boldsymbol{r})$ represents the matter density at the coordinate point $\boldsymbol{r}$. On the other hand, the vector form of equation (1) must necessarily be of the form

$$\boldsymbol{a} = a_0 f(x) \boldsymbol{e}_a, \tag{23}$$

where the unit vector $\boldsymbol{e}_a$ points in the direction of the acceleration $\boldsymbol{a}$. Substitution of equation (23) into (22) with help of relation (5) yields

$$x^2 \boldsymbol{e}_a = -\frac{G}{a_0} \int \frac{(\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \, \rho(\boldsymbol{r}') \mathrm{d}V'. \tag{24}$$

This description generalises the previous relations in the sense that a point mass M can be directly substituted for $M(r)$ in all relevant equations.

The magnitude of the vectorial equation (24) is the formula to calculate the variable x for a given mass distribution density $\rho(\boldsymbol{r})$. As an example, for a point mass M the density is given by $\rho(\boldsymbol{r}) = M\delta(\boldsymbol{r})$, where δ represents Dirac's delta function. Consequently, the variable x is given by equation (2) as expected. For the case of a spherically symmetric distribution of matter, the density depends only on the radial coordinate r. This means that (Binney & Tremaine 2008) $x^2 \boldsymbol{e}_a = -GM(r)\boldsymbol{e}_r/r^2$, where $M(r)$ is the mass contained within the radius r and $\boldsymbol{e}_r$ is a unit vector in the direction of the radial coordinate. With this, it follows that the variable x depends only on the mass contained within the radius r. Since the general acceleration function $f(x)$ is analytic (cf. equation (6)), then it is also clear from this point of view that Newton's theorems are valid for the class of models presented in this article.

Recently Milgrom (2010) has developed a quasi-linear formulation of MOND, which in particular can by applied to our spherical symmetric case (see also Zhao & Famaey 2010). We now show that this is equivalent to our formulation. In this theory, a general gravitational potential $\Phi := \phi_{\mathrm{N}} + \varphi$ is proposed, where φ satisfies the equation

$$\nabla^2 \varphi = \nabla \cdot \left[\nu \left(|\nabla \phi_{\mathrm{N}}| / a_0 \right) \, \nabla \phi_{\mathrm{N}} \right], \tag{25}$$

where $\nu(y)$ represents a new interpolation function which tends to $y^{-1/2}$ far away from the strongest gravity regime. Our approach is equivalent to their results, since the connection between $f(x)$ and ν is given by

$$f(x) = x^2 \left[\nu(x^2) + 1 \right]. \tag{26}$$

Comparing this last result with equation (5) it follows that $\mu = (\nu + 1)^{-1}$ and so, for $\nu \gg 1$, then $\mu = \nu^{-1}$. For the specific value $\nu(y) = y^{-1/2}$ then

$$f(x) = x + x^2. \tag{27}$$

The above two equations show that the function f is specifically a function of x^2 only, i.e. it depends exclusively on the Newtonian force x^2.

In the following sections, we discuss several astrophysical systems where the proposed gravity law (20) can be tested over a wide range of values of x, and explore the various predictions which emerge. We perform comparisons mostly through expected scalings related to velocity dispersions, masses and equilibrium radii, derived from very general dimensional arguments. We consider mostly spherical mass distributions $M(r)$, making use of the validity of Newton's theorems for spherical symmetry already discussed.

2.1 Solar system consistency and an illustrative rotation curve of our Galaxy

As a first test of the proposed force law we compare the variations with respect to Newtonian acceleration which would be introduced at solar system scales, to the exquisitely measured upper limits on this quantity.

For solar system scales with distances of between 0.1 and ~ 50 AU and taking $M = 1\,\mathrm{M}_{\odot}$, we get $10^2 \lesssim x \lesssim 10^4$ (cf. Figure 7). Due to these large values, we note immediately that the deviations from Newtonian dynamics are negligible. Figure 3 shows that our model lies well within the observational upper limits for the departures from Newtonian radial acceleration for the planets in our solar system, as reported by Sereno & Jetzer (2006). In consequence, under the proposed force law, MONDian type dynamics have no relevance at solar system scales. In light of our results, we argue in favour of studies by Anderson et al. (2002); Turyshev & Toth (2009); Toth & Turyshev (2009), that regard the anomaly of the recession velocity of the Pioneer probes as an effect of the thermal radiation of the spacecrafts (see however, Bekenstein 2006; Milgrom 2009a, for interpretations of the anomaly as a signature of MOND).

As a particular example, we now turn to the rotation velocity curve of our Galaxy. The circular rotation velocity $V(r)$ associated to equation (20) is given by

$$V(r) = \left(a_0 l_{\mathrm{M}} \, \frac{1 - x^4}{1 - x^3} \right)^{1/2}. \tag{28}$$

For the case of our Galaxy, the mass of the bulge and the mass of the disc as a function of the radial distance are respectively given by (see e.g. Allen & Santillan 1991)

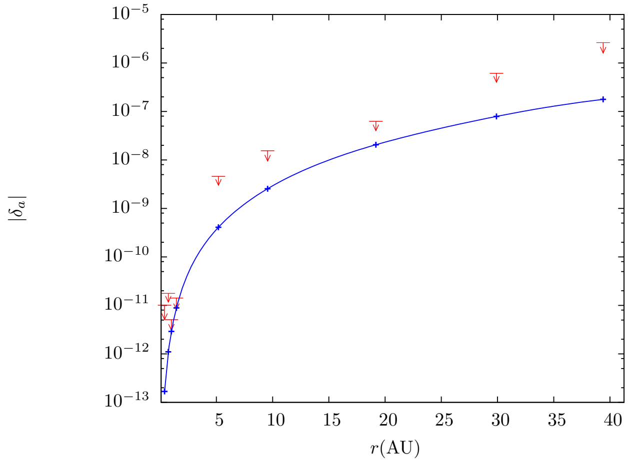

Figure 3. The curve shows the fractional modifications $\delta a := (a - a_{\mathrm{Nt}})/a_{\mathrm{Nt}}$ our model introduces when compared to purely Newtonian acceleration a_{Nt}. This is compared with the upper limits on the deviations from radial Newtonian acceleration as reported by Sereno & Jetzer (2006). The diagram shows that the modifications introduced by our model are always within the observed upper limits. From left to right the crosses represent the calculated values δa at the distance of Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune and Pluto respectively.

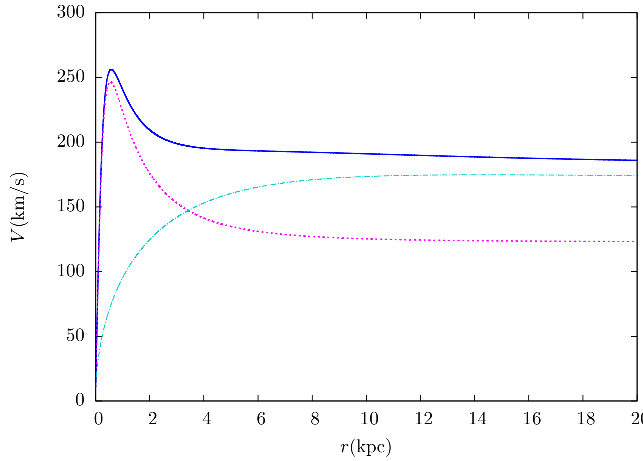

Figure 4. The figure shows the rotation curve (continuous line) of our Galaxy using the theory of gravity proposed in this article (cf. equation (20)). We have assumed that only disc (dashed-dotted line) and bulge (dotted line) contributions to the rotation curve are present.

$$m(r)\big|_{\mathrm{bulge}} = m\, r^3 \, \left(a^2 + r^2\right)^{-3/2}, \tag{29}$$

$$m(r)\big|_{\mathrm{disc}} = 2\pi\Sigma_0 r_\star^2 \left[1 - \left(\frac{r}{r_\star} + 1\right) e^{-r/r_\star}\right], \tag{30}$$

where $m = 1.4 \times 10^{10}\,\mathrm{M}_\odot$, $a = 0.387\,\mathrm{kpc}$, $\Sigma_0 = 7 \times 10^8\,\mathrm{M}_\odot\,\mathrm{kpc}^{-2}$ and $r_\star = 3.5\,\mathrm{kpc}$. The effective mass at a distance r from the centre of our Galaxy is then given by

$$M(r) = m(r)\big|_{\mathrm{bulge}} + m(r)\big|_{\mathrm{disc}}, \tag{31}$$

and so, by substitution of equation (31) into (28) the theoretical rotation curve of our Galaxy, shown in Figure 4, is obtained. Given the validity of Newton's theorems for spherical mass distributions in our case, the inner rotation curve of Figure 4 is fully self consistent. This occurs since in that region, the dynamics are completely dominated by the galactic bulge, which has the same spherical symmetry as is assumed in the calculation. The same holds to a large degree for the calculated rotation curve beyond a few disc scale radii, where the mass distribution has essentially converged. In the intermediate regime, the assumption of spherical symmetry in the calculation necessarily introduces some error. However, by analogy with the Newtonian case, where the rotation curve of an exponentially decreasing infinitely thin disc deviates only about 20% from that of the corresponding spherical mass distribution (see e.g Binney & Tremaine 2008), this error is probably small. The validation of this analogy through a full 3D implementation of our model, lies beyond the scope of this first presentation of our model. In the light of this, Figure 4 simply provides a graphical representation of the fact that our $f(x)$ was calibrated from our Galaxy's best fit $\mu(a/a_0)$ as given by Famaey & Binney (2005).

This rotation curve represents well the observed features of our Galaxy's rotation curve (see e.g Kuijken & Gilmore 1989). This was actually built into our model, as the value of n in the proposed force law was calibrated from the numerical MOND interpolation function estimated by Famaey & Binney (2005), which was obtained precisely from calibrations against the rotation curve of our Galaxy.

3 EQUILIBRIUM RADII OF BOUND CONFIGURATIONS

If we think of an arbitrary astrophysical equilibrium structure with total mass M, with characteristic equilibrium radius r_{e} and internal velocity dispersion σ, we can estimate the relation between the above three quantities by equating the kinematic pressure to the mean gravitational force. Note that in this section we use M to denote the total mass of an extended astrophysical system. Writing the kinematic pressure per unit mass as $C\rho\sigma^2/M$, replacing ρ for Mr_{e}^{-3} and equating this to the proposed force law of equation (20) we obtain

$$C\tilde{\sigma}^2 = \frac{R_{\mathrm{e}}^4 - 1}{R_{\mathrm{e}}^4 - R_{\mathrm{e}}}, \tag{32}$$

where we have introduced a dimensionless velocity dispersion $\tilde{\sigma} := \sigma/(Ga_0M)^{1/4}$ and an equilibrium radius $R_{\mathrm{e}} := r_{\mathrm{e}}/l_{\mathrm{M}}$ for the problem. The constant C is expected to be of order 1.

We notice that for small values of R_{e}, corresponding to large values of the variable x, i.e. the Newtonian regime, the right hand side of equation (32) reduces to $1/R_{\mathrm{e}}$. In this case equation (32) leads directly to the virial equilibrium relation of Newtonian gravity $r_{\mathrm{e}} = C^{1/2}\sigma/(G\rho)^{1/2}$. At the opposite MONDian limit, with small values of x and so, large values of R_{e}, the right hand side of equation (32) tends to 1. In this limit, equation (32) yields the Faber-Jackson re-

lation $C^{1/2}\sigma = (Ga_0 M)^{1/4}$. Allowing for a proportionality between isotropic velocity dispersions in pressure supported systems and rotation velocities in angular momentum supported ones, this last relation could also be seen as the baryonic Tully-Fisher relation which also has an index close to 4 (see e.g. Puech et al. 2010). It is interesting that in terms of the equilibrium gravitational radii of the proposed force law, the galactic Faber-Jackson relation appears as the "Jeans mass" solution for large values of R_{e}. Intermediate cases should be well described by the appropriate points along the transition between the two limits of equation (32).

If we think of a gravitationally bound system with an associated dispersion velocity σ, the above results can also be easily understood from the point of view of dimensional analysis. Indeed, employing the same procedure we used in constructing equation (1), the problem is now characterised by the dispersion velocity σ of the system, the radius r, the total mass M, Newton's constant of gravity G and Milgrom's constant a_0. With these, Buckingham's theorem of dimensional analysis demands that the velocity dispersion must have the following form

$$\tilde{\sigma} = g(x), \tag{33}$$

with $g(x)$ an arbitrary function to be determined. A natural approximation often used, is to model observations over a limited parameter range through power law representations. As such, we can explore the consequences of imposing a power law for the function $g(x)$, i.e.,

$$\tilde{\sigma} \propto x^{\alpha} = l_{\mathrm{M}}^{\alpha} r^{-\alpha}. \tag{34}$$

By comparing equations (32) and (34), we see that equation (34) is merely a power law approximation to the full relation supplied by the proposed force law. One is lead to expect that over any limited range of values of x, equation (32) should be accurately approximated by equation (34), with a suitable choice of the parameter α, which will more generally be a function of x.

The strong prediction of equation (20), as seen in equation (34), is that over any constrained range of values of x, e.g. corresponding to any well defined class of astrophysical objects, a relation of the type

$$\sigma = C r^{\beta} M^{\gamma}, \tag{35}$$

will always appear. Comparison with equation (34) yields directly $\beta = -\alpha$ and shows that the two power law indices of equation (35) are not independent, but will obey the necessary constraint

$$\gamma = \frac{1}{4} - \frac{\beta}{2}. \tag{36}$$

Again, we see that the two limits explored above correspond to $\alpha = 1/2$ and $\alpha = 0$, for the Newtonian and deep MOND regimes respectively. It is clear that the constraint of equation (36), obtained through dimensional analysis, is satisfied in these two cases. Note that this constraint can also be derived through a power law approximation to the full force model of equation (32). In what follows it will become evident that this power law approximation and the corresponding predicted constraint (36) accurately reproduce the empirical scalings observed in elliptical galaxies and the well studied dwarf spheroidals of the local group. We shall also present an exact expression for $\alpha(x)$ at all scales as a strong testable prediction.

3.1 The fundamental plane of elliptical galaxies

The clearest correlations between the structural parameters of elliptical galaxies are expressed by the fundamental plane relations, first described by Djorgovski & Davis (1987); Dressler et al. (1987), as scaling relations between the observed effective radius r of the Galaxy, the central line of sight velocity dispersion σ, and the mean surface brightness I within the effective radius (see e.g. Bernardi et al. 2003; Desroches et al. 2007, for recent observations). Assuming a constant mass to light ratio and using the fact that the mean surface brightness is given by the luminosity L within the effective radius, such that $I = L/2r^2$, measurements over large samples, now in the thousands of galaxies, give a relation identical to equation (35).

In what follows we present a treatment for elliptical galaxies in isolation. Many such systems reside in galaxy clusters where external field effects might have some relevance, e.g. in accounting for the escape of high velocity stars (Wu et al. 2008). However, the well established constancy of the basic scaling relations for ellipticals across different density environments, implies that these external field effects are by far not the physical causes driving the structure of the fundamental plane. As such, we shall not consider such environmental effects further at this point.

A broad agreement in the literature yields values close to the ones reported by Bernardi et al. (2003) of $\beta = -0.3356 \pm 0.023$ and $\gamma = 0.503 \pm 0.025$. The first interesting point to note is that the above measured values deviate for β, slightly but noticeably, from the Newtonian expectations of $\beta = -1/2, \gamma = 1/2$. This feature is termed the tilt in the fundamental plane, and has traditionally been explained in terms of mass to light ratios which vary as a function of mass for elliptical galaxies. Although such explanations are not unreasonable (see e.g. Desroches et al. 2007), we shall see that this tilt in the fundamental plane can be accounted for, at constant mass to light ratios, by the slight deviations from the Newtonian limit expected from equation (32) at the values of R_{e} which correspond to elliptical galaxies.

As already seen from equation (32) under the power law approximation of equation (35), we can compare the observed values for the power law scalings of the fundamental plane of equation (35) against the theoretical expectations of our model through equation (36). For the observed value of $\beta = -0.3356 \pm 0.023$, equation (36) implies $\gamma = 0.418 \pm 0.012$. This is marginally consistent with recent measurements of $\gamma = 0.503 \pm 0.025$ at a two sigma level, under the hypothesis of a constant average mass to light ratio for elliptical galaxies. We therefore see that a power law approximation through a dimensional analysis approach to the problem furnishes the relation between the indices β and γ in equation (35), given by relation (36). The actual value of both indices will be obtained theoretically through the use of the full force model in what follows. It is noteworthy that practically all of the tilt in the fundamental plane can be

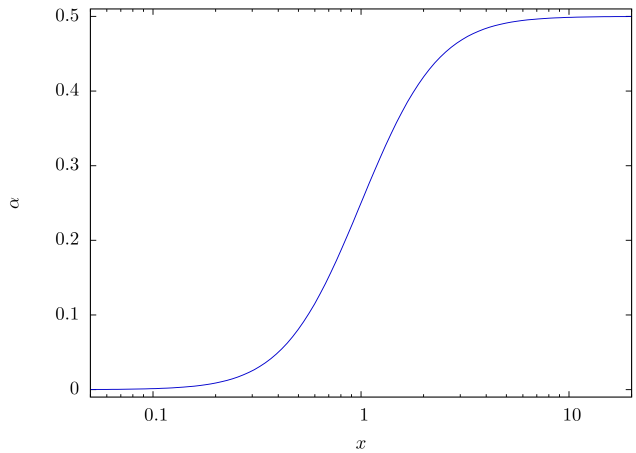

Figure 5. The plot shows the variation of the power index α with respect to x, in the general gravitational equilibrium relation (see text) valid for all systems and at all scales. For large values of the parameter x, the Newtonian limit $\alpha = 1/2$ is reached, and for small ones convergence to the MONDian one $\alpha = 0$ is obtained. As shown in Figure 7 many astrophysical systems of interest fall in the intermediate regions about $x = 1$.

accounted for directly by the expectations of equation (36). The fullest explanation might well include a certain degree of mass dependence for the average mass to light ratios, as pointed out by various studies (see e.g Desroches et al. 2007).

In order to evaluate the parameter α we proceed by fitting a power law function of the form (34) to equation (32). This can be done locally to the smooth function (32) by calculating the tangent line to the function $\log \tilde{\sigma}(\log x)$ at a given point $P_0(x_0, \tilde{\sigma_0})$, which satisfies the relation

$$\log \tilde{\sigma} = \alpha(x_0) \log x + \log \tilde{\sigma_0} - \alpha(x_0) \log x_0. \qquad (37)$$

The slope of the tangent line is given by

$$\alpha(x) = \frac{\partial \log \tilde{\sigma}}{\partial \log x} = \frac{x^3}{2} \frac{3 + 2x + x^2}{(1 + x + x^2)(1 + x + x^2 + x^3)}, \qquad (38)$$

which simply means that the constant α depends on the scale x_0 of the system as shown by Figure 5. Notice that $\alpha \to 1/2$ for the Newtonian $x \to \infty$ regime and $\alpha \to 0$ for the MONDian $x \to 0$ limit.

For any given class of astrophysical objects, the parameter α can now be obtained by evaluating x and then solving for α from equation (38).

Note that typical values for elliptical galaxies with radii of order a few kpc, and masses one order of magnitude around $10^{10} \mathrm{M}_\odot$ (see Figure 7), we get $x \sim 1$, which from equation (38) yields $\alpha \sim 0.3$. The tilt in the fundamental plane is hence naturally explained by the force law presented, requiring only minimum departures from constant average values for the mass to light ratios of elliptical galaxies.

Recent studies spanning larger ranges of velocity dispersion and mass values for elliptical galaxies (e.g. Gargiulo et al. 2009; Desroches et al. 2007), have begun to identify slight trends in the power indexes of the fundamental plane. This is exactly what one would expect from the developments presented here, as when one broadens the range of physical parameters, the power law approximation to equation (32) ceases to be valid, and a warp will necessarily develop. Being elliptical galaxies very close to the Newtonian limit of $\alpha = 1/2$, we can estimate the first order trend of this warp against mass by considering that typical densities for elliptical galaxies tend to drop as one goes to more massive systems, which would move the fit further away from the Newtonian regime, leading to a gradual decrease in α with increasing total mass. By comparing with Figure 8 of Gargiulo et al. (2009), we see precisely this very trend, with the measured value for their index a in the $\log r_\mathrm{e} = a \log \sigma_0$ fit (notice that $a \propto \alpha^{-1}$) increasing as the limit mass of the galaxies included in the sample increases. Therefore, the measured first order trends for the fundamental plane indexes with mass, are seen to agree with the expectations of the model presented.

3.2 Local dwarf spheroidal galaxies

We now turn to the well studied dSph galaxies of the local group and their scaling relations. Taking this time the first order terms for $R_\mathrm{e} \gg 1$ in equation (32), we obtain

$$C\tilde{\sigma}^2 = 1 + \frac{1}{R_\mathrm{e}^3}. \qquad (39)$$

Again we see that the first deviations from the deep MOND regime occur after skipping the R_e^{-1} and R_e^{-2} terms. This implies that near the deep MOND regime, gravitational physics remain largely unaltered. If we now solve for r_e from the above equation, we obtain

$$r_\mathrm{e}^3 = \frac{(GMa_0)^2}{a_0^3 \left[C\sigma^2 - (GMa_0)^{1/2} \right]}. \qquad (40)$$

Through full integration of isothermal equilibrium density profiles for the local dSph's under a force law equivalent to what we propose here in the deep MOND regime, Hernandez et al. (2010) showed that a suitable description of the problem can be found without the need to invoke the presence of any dark matter. Still, this might require the consistent inclusion of the disruptive effects of galactic tides on the local dwarfs, as pointed out by e.g. Sánchez-Salcedo & Hernandez (2007); Angus (2008); McGaugh & Wolf (2010). It is interesting to remark that Hernandez et al. (2010) also showed that there is a very clear correlation between the mass to light ratio resulting from the dynamical modelling in MOND type prescriptions (or alternatively, the dark matter fraction for these systems), and the relative ages of the stellar populations present, as directly inferred from statistical studies of their resolved stellar populations (see e.g. Hernandez et al. 2000; Dolphin 2002; Helmi 2008; Martin et al. 2008; Tolstoy et al. 2009). This correlation is natural in a scenario where only the stars present act as a source of gravity, shining less as they age, but appears contrived under the dark matter hypothesis. Also, Hernandez et al. (2010) showed the expected scaling on average between σ^4 and total stellar mass for the objects in question, within observational errors, expected from the deep MOND regime. Introducing this scaling into equation (40) we obtain

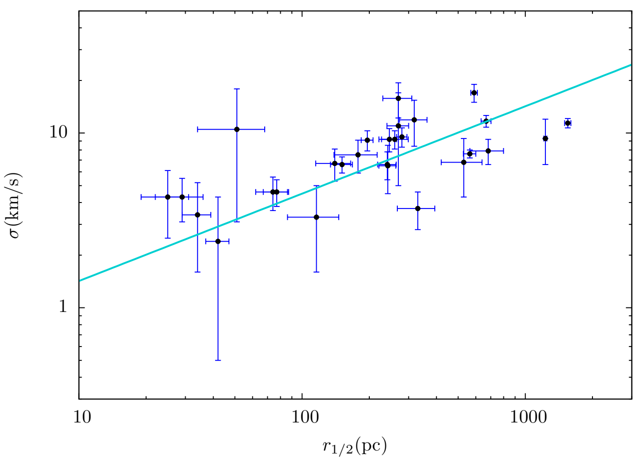

Figure 6. Measured values of half light radii $r_{1/2}$ and internal velocity dispersions σ for the local dSph galaxies with their associated one sigma uncertainties. The solid line gives a best fit $\sigma^2 \propto r_{1/2}$ relation, the exponent of which results from the proposed force law. Data from the compilation of Walker et al. (2010). The point for the heavily distorted Sagittarius dwarf shown at the right end has been excluded from the fit.

$$r_e \propto \sigma^2. \tag{41}$$

We can now test the expectations of our proposed force law through equation (41), by plotting the measured values of the velocity dispersion for the local dSph galaxies, against their observed half-light radii. We assume that our r_e in equation (41) will be equal to some constant times the measured half light radius $r_{1/2}$ and take the compilation for these values found in Walker et al. (2010) to plot Figure 6. We see that in spite of the large errors present, the expected scaling of $\sigma^2 \propto r_{1/2}$ given by the line shown, clearly holds rather well. We have preferred this particular plane to test our model, as velocity dispersions and half light radii are the directly measured observables, not affected by the assumption of rather uncertain mass to light ratios.

Finally, we can try a first order extrapolation from the smallest $R_e \ll 1$ systems, the dSph galaxies, to the largest astrophysical structures in that regime, galaxy clusters, by taking values of $r = 1$kpc and $\sigma = 10\,\mathrm{km\,s^{-1}}$ from Figure 6 and increasing the velocity dispersion by a factor of 50. The expectation of equation (41) then being radii of 2.5×10^3 times larger. Indeed, 2.5 Mpc and $500\,\mathrm{km\,s^{-1}}$ are representative values for the structural parameters of large clusters of galaxies. Explaining the dynamics of galaxy clusters in the pure MOND formalism requires some unseen dark matter (see e.g. Sanders 2003; Angus et al. 2010), so we defer the details of this problem to a latter study. Here we only point out that the first order scaling of equation (41), extrapolated across ~ 6 orders of magnitude in radius, is at least not significantly in error.

Again, we see that having a well defined force law, rather than dealing with a cumbersome MOND interpolation function, facilitates the physics considerably. This allows a more direct tracing of the astrophysical consequences, and yields more transparent predictions. Notice that the general results of the whole Section 3 are not dependent on the details of $f(x)$, and will hold for any such function within the theoretical framework introduced. We note however, the good agreement with observations for the particular $f(x)$ used.

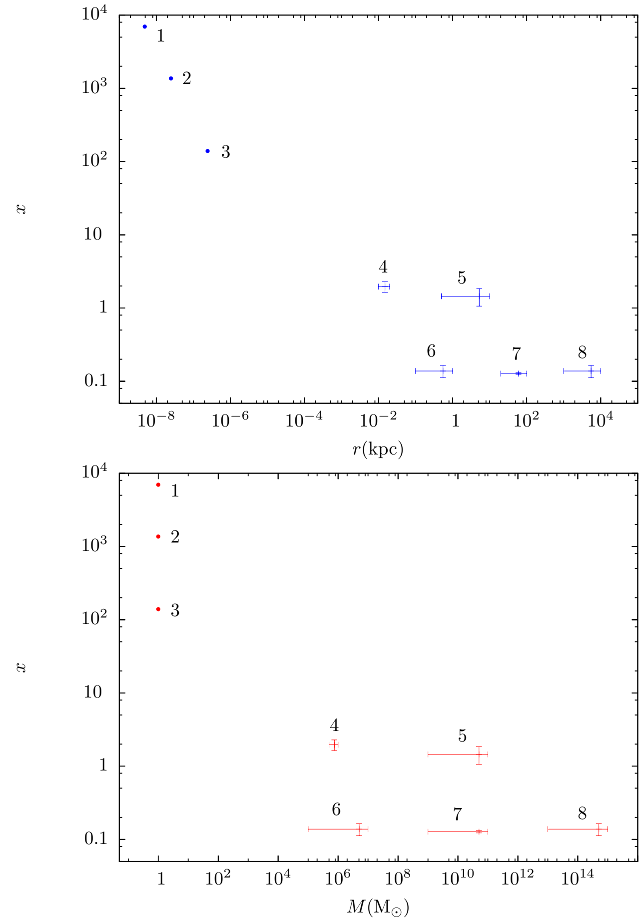

Figure 7. The figure shows different values of $x \propto M^{1/2}/r$ for typical masses M and radii r of different astrophysical systems: (1) Solar system at Earth's orbit. (2) Solar system at Jupiter's orbit. (3) Solar system at the Kuiper belt radius. (4) Globular clusters. (5) Elliptical galaxies and bulges of spirals. (6) Dwarf spheroidal galaxies. (7) Outer regions of spiral galaxies. (8) Galaxy clusters.

As a final summary of this section, we present Figure 7, with typical values of the parameter x for average masses and radii for a range of astrophysical objects. This Figure, when compared to Figure 5 clearly complements the results of the previous sections. We see that solar system dynamics will be utterly unaffected by the modification proposed. Marginal modifications will appear in the dynamics of globular clusters (to within observational uncertainties, see e.g. Hernandez et al. 2010). A slight tilt is to be expected in the fundamental plane of elliptical galaxies and we see that the external regions of spiral galaxies, galaxy clusters and local dwarf spheroidal galaxies should appear as the most heavily "dark matter dominated" systems.

4 CONCLUDING REMARKS

We have shown in this article that if a modification of either the dynamical or the gravitational part of Newton's gravity is to be performed, a gravitational modification has a greater advantage for physical clarity and ease of calculations. Furthermore, it has become clear that with the addition of Milgrom's acceleration constant a_0, all physical relations must be at least functions of the parameter x defined in equation (2), according to Buckingham's theorem of dimensional analysis. As examples, we have shown that the acceleration felt by a particle on a centrally symmetric gravitational field satisfies this condition. Also, we show that the velocity dispersion associated to a system in the linear approximation gives rise to a generalised gravitational equilibrium relation which converges to the well known results of the fundamental plane in the description of elliptical galaxies, the Tully-Fisher relation for spiral galaxies, the Faber-Jackson relation for elliptical galaxies, the Jeans' stability criterion in the Newtonian limit and to corresponding observed relations for dSph's galaxies.

The inclusion of a_0 in the theory of gravity means that a characteristic mass-length l_{M} has been added. This scale approximately indicates the transition between the MONDian and Newtonian regimes. By a suitable choice of model, this transition can be built smoothly. On very general grounds, a characteristic scale l_{M} has been introduced into gravitation and so, gravity is no longer scale invariant.

We note also another testable prediction: as precision increases at solar system measurements the upper limits to departures from Newtonian acceleration will converge to definite values, as implied by Figure 3.

The particular acceleration function we built is perhaps only an approximation to a more general gravitational law. The functions we built, although logical and quite precise in principle, may just be good candidates for the real gravitational theory and they should be thought of as such: good approximations which fit a wide variety of available astrophysical data to within errors. However, it has to be through an extension of the general theory of relativity that a more precise and fundamental form is to be found. This extended relativistic theory of gravity must show that the PPN approximations converge in some limit to the acceleration function built in this article.

The Newtonian character of gravity has not been convincingly proven across the astrophysical scales explored in this article and as such, modifications to the theory are feasible. It is also important to note that a very wide variety of modified gravity theories of the type developed here, including MOND, are fundamentally falsifiable: a single $a > a_0$ system appearing as heavily dark matter dominated, would invalidate them all as an alternative to the dark matter hypothesis.

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